Size Ramsey numbers of forests versus double stars and brooms

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Outline

1. Introduction
   - Size Ramsey number
   - Double stars
   - Brooms
2. Main results
   - Matchings versus double stars and brooms
   - P_3 versus double stars and brooms

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Basic definition

- We assume all graphs are finite, undirected and simple. For a graph G, the vertex set, edge set, order and size of G are denoted by $V(G)$, $E(G)$, $v(G)$ and $e(G)$, respectively. The *neighborhood*, $N_G(v)$ or $N(v)$ for short, of a vertex v of G is the set of vertices adjacent to it. The *degree*, $\deg_G(v)$ or $\deg(v)$ for short, of a vertex v of G is the number of edges incident to it. The *minimum degree* of G, $\delta(G)$, is the smallest of the degrees of vertices in G and the *maximum degree*, $\Delta(G)$, of G is the largest of the degrees of the vertices in G. For any subset X of $V(G)$, let $G[X]$ denote the subgraph induced by X; similarly, for any subset F of $E(G)$, let $G[F]$ denote the subgraph induced by F. We use $G - X$ to denote the subgraph of G obtained by removing all the vertices of X together with the edges incident with them from G. If $X = \{v\}$, we simply write $G - v$ for $G - \{v\}$.

Size Ramsey number

- If G and H are two graphs, we write $F \rightarrow (G, H)$ to denote that G or H is a monochromatic subgraph of F in every 2-coloring of the edges of F. The *size Ramsey number* $\hat{r}(G, H)$ of two graphs G and H, introduced by Erdös et al., is defined as

$$\hat{r}(G, H) = \min\{e(F) : F \rightarrow (G, H)\}.$$

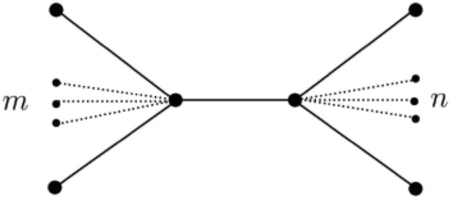
Basic proof idea

- Upper bound: Construct a graph F such that $F \to (G, H)$, then $e(F)$ is the upper bound of $\widehat{\mathrm{r}}(G, H)$.
- Lower bound: Any graph with a given number of edges can find a red/blue edge-coloring such that these graphs contain neither a red copy of G nor a blue copy of H.

Outline

$D(m, n)$

A *double star* is a graph with two stars and an edge connecting their centers.



$v(D(m,n)) = n+m+2$, $e(D(m,n)) = n+m+1$, $\Delta(D(m,n)) = \max\{n+1, m+1\}$.

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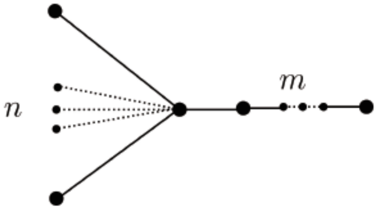
$B(m, n)$

A *broom* is a graph which is a path with a star at one end.



$$v(B(m,n)) = n+m+1,\ e(B(m,n)) = n+m,\ \Delta(B(m,n)) = n+1.$$

Outline

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The subgraph induced by the red edges

Fact 1

To avoid red $2P_2$, the subgraph induced by the red edges in F is a star or a triangle.

$\widehat{\mathrm{r}}(sP_2, D(m,n))$

Theorem 2.1

For $n \geq m \geq 2$ and $s \geq 2$,

$$s(n+1)+m \leq \widehat{\mathrm{r}}(sP_2, D(m,n)) \leq s(n+m+1).$$

Upper bound of $\widehat{r}(sP_2, D(m, n))$

Construct a graph F such that $F \to (sP_2, D(m, n))$:

$$sD(m, n)$$

$$e(sD(m, n)) = s(n + m + 1).$$

Lower bound of $\widehat{r}(sP_2, D(m,n))$

Let F be a graph of size at most $s(n+1)+m-1$ and containing a copy of $D(m,n)$. Next use induction on s:
Assume $s=2$.

- $\Delta(F) \leq n$: no $D(m,n)$ copy.
- $\Delta(F) \geq n+1$: Take a vertex v of maximum degree, and color the edges incident to v red and the other edges blue.

Suppose the conclusion holds for $s-1$. Again, take a vertex v of maximum degree. Denote $F' = F - v$. Then

$$e(F') \leq s(n+1)+m-1-(n+1) = (s-1)(n+1)+m-1.$$

Then by inductive hypothesis, there is a red-blue coloring of $E(F')$ such that there is neither a red copy of $(s-1)P_2$ nor a blue copy of $D(m,n)$. Now keep this coloring, and further color the edges red incident to v, then we get the red-blue coloring which we want.

$\widehat{\mathrm{r}}(2P_2, D(1, n))$

Theorem 2.2

For $n \geq 3$,

$$\widehat{\mathrm{r}}(2P_2, D(1, n)) = 2n + 4.$$

Upper bound of $\widehat{r}(2P_2, D(1, n))$

By Theorem 2.1, we have $\widehat{r}(2P_2, D(1, n)) \leq 2n + 4$.
Next we show $\widehat{r}(2P_2, D(1, n)) \geq 2n + 4$.

Lower bound of $\widehat{r}(2P_2, D(1, n))$

Let F be a graph of size $2n+3$ and containing a copy of $D(1, n)$.
Take $w \in V(F)$ such that $\deg(w) = \Delta(F)$.
Let $F' = F - w$.

- $\Delta(F) \leq n$: no $D(1, n)$ copy.
- $\Delta(F) \geq n+2$: $e(F') < e(D(1, n))$, thus no $D(1, n)$ copy.
- $\Delta(F) = n+1$: need further discussion.

Lower bound of $\widehat{r}(2P_2, D(1, n))$

Thus assume that $\Delta(F) = n + 1$.
Then $e(F') = n + 2 = e(D(1, n))$.

- $F' \ncong D(1, n)$: we have done.
- $F' \cong D(1, n)$: need further discussion.

Lower bound of $\widehat{r}(2P_2, D(1, n))$

Assume $F' \cong D(1, n)$.
Let u and v be the n-center and 1-center, respectively.

- $N(w) \cap V(F') = \emptyset$: color one edge of F' red and the other edges blue.

Thus assume $N(w) \cap V(F') \neq \emptyset$.

- $v \in N(w) \cap V(F')$: color the edges incident to v red and color the other edges blue.

Lower bound of $\widehat{r}(2P_2, D(1, n))$

Thus assume $v \notin N(w) \cap V(F')$.

- $u \in N(w) \cap V(F')$: color the edge uw red and color the other edges blue.

Thus assume $u \notin N(w) \cap V(F')$.

- $\exists x \in N_{F'}(u)$ such that $x \in N(w) \cap V(F')$: color ux and wx red and color the other edges blue.

Thus assume no neighbor of u belong to $N(w) \cap V(F')$.

- Denote by v_1 the other neighbor of v than u in F' such that $N(w) \cap V(F') = \{v_1\}$: color vv_1 and wv_1 red and other edges blue.

And such coloring is what we want.

$\widehat{r}(sP_2, D(1, n))$

Next by an induction on s, we have the following result:

Lemma 2

Assume F is a graph of size $s(n+2)-1$ where $s \geq 2$ and $n \geq \lceil (s^2+3s-2)/2 \rceil$. Denote by W the set of vertices of degree at least $n+1$ in F. If $|W| \geq s$, then $|W| = s$.

Theorem 2.3

For $s \geq 2$ and $n \geq \lceil (s^2+3s-2)/2 \rceil$,

$$\widehat{r}(sP_2, D(1, n)) = s(n+2).$$

Skipping the proof.

$\widehat{r}(2P_2, D(2,2))$

Theorem 2.4

$$\widehat{r}(2P_2, D(2,2)) = 10.$$

Skipping the proof.

Problem of $\widehat{r}(2P_2, D(n, n))$

Problem 2.1

For $n \geq 3$,

$$\widehat{r}(2P_2, D(n, n)) = ?.$$

$\widehat{\mathrm{r}}(2P_2, B(m,n))$

Theorem 2.5

For $m \geq 4, n \geq 1,$

$$\widehat{\mathrm{r}}(2P_2, B(m,n)) \leq 2n + 2m - 2.$$

For $m \geq 3, n \geq m+2,$

$$\widehat{\mathrm{r}}(2P_2, B(m,n)) \geq 2n + m + 2.$$

Upper bound of $\widehat{r}(2P_2, B(m,n))$

Construct a graph F such that $F \to (2P_2, B(m,n))$:
let F be a graph obtained from a $2m$-cycle $v_1 v_2 \ldots v_{2m}$ and two stars $K_{1,n-1}, K_{1,n-1}$ with centers u, v, respectively, by identifying u_1, v_1 and u_2, v_m (that is $u_1 = v_1$ and $u_2 = v_m$), which show in Figure 1(a).

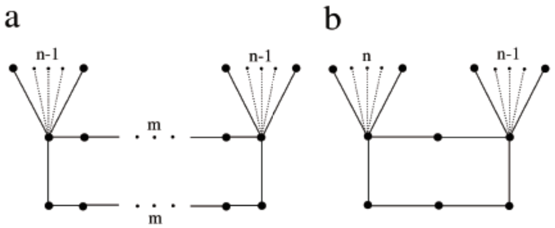


Figure 1: Graphs for Theorem 2.5 and Corollary 2.1.

Lower bound of $\widehat{r}(2P_2, B(m,n))$

Let F be a graph with at most $2n+m+1$.
We can assume that $e(F) = 2n+m+1$.
Take $w \in V(F)$ such that $\deg(w) = \Delta(F)$.
Let $F' = F - w$.

- $\Delta(F) \leq n$: no $B(m,n)$ copy.
- $\Delta(F) \geq n+2$: $e(F') < e(B(m,n))$, thus no $B(m,n)$ copy.
- $\Delta(F) = n+1$: need further discussion.

Lower bound of $\widehat{r}(2P_2, B(m, n))$

Suppose $\Delta(F) = n+1$. Let s be the number of vertices of degree $n+1$.

Claim 3

$$s \leq 2.$$

Proof.

Assume, to the contrary, that $s \geq 3$. Then there exist three vertices, say u_1, u_2, u_3, such that $deg_F(u_i) = \Delta(F) = n+1$ for $i = 1, 2, 3$, and hence there are at least $3(n+1) - 3 \geq 2n + m + 2 > e(F)$, since $n \geq m+2$, a contradiction. ∎

Lower bound of $\widehat{r}(2P_2, B(m,n))$

Suppose $s = 2$. Then there exist two vertices $u, v \in V(F)$ such that $deg_F(u) = deg_F(v) = \Delta(F) = n + 1$.

- $uv \in E(F)$: color the edge uv red, and then color the other edges blue.
- $uv \notin E(F)$ and $N_F(u) \cap N_F(v) \neq \emptyset$, then $\exists x \in N_F(u) \cap N_F(v)$: color the edges ux, vx red, and then color the other edges blue.
- $uv \notin E(F)$ and $N_F(u) \cap N_F(v) = \emptyset$: note that $e(F - u - v) = m - 1$ and $F - v \cong B(m,n)$, $F - u \cong B(m,n)$, then these $m - 1$ edges must form a path. We color one edge from this path red, and then color the other edges blue.

Lower bound of $\widehat{r}(2P_2, B(m, n))$

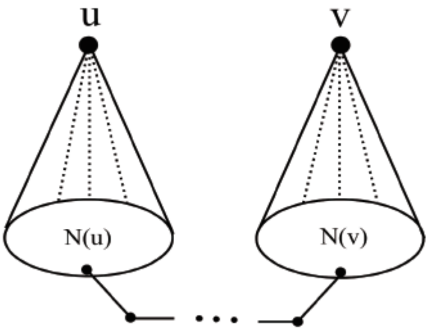


Figure 2: Graph for the Proof of Theorem 2.5.

$\widehat{\mathrm{r}}(2P_2, B(3,n))$ and $\widehat{\mathrm{r}}(2P_2, B(4,n))$

Corollary 2.1

For $n \geq 5$,

$$\widehat{\mathrm{r}}(2P_2, B(3,n)) = 2n+5.$$

Corollary 2.2

For $n \geq 6$,

$$\widehat{\mathrm{r}}(2P_2, B(4,n)) = 2n+6.$$

Outline

The subgraph induced by the red edges

Fact 4

To avoid red P_3, the subgraph induced by the red edges in F is a matching.

$\widehat{r}(P_3, D(m, n))$

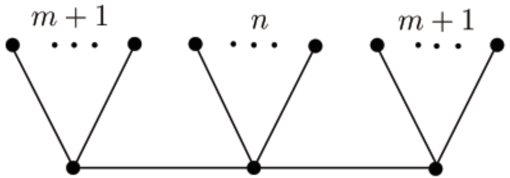
Theorem 2.6

For $n \geq m \geq 2$,

$$n + 2m + 1 \leq \widehat{r}(P_3, D(m, n)) \leq n + 2m + 4.$$

Upper bound of $\widehat{r}(P_3, D(m, n))$

Construct a graph F such that $F \to (P_3, D(m, n))$:
Let uvw be a path of order 3, and F be a graph obtained from three stars $K_{1,m+1}$, $K_{1,n}$ and $K_{1,m+1}$ by identifying the center of $K_{1,m+1}$ and u, the center of $K_{1,n}$ and v and the center of $K_{1,m+1}$ and w, respectively.



Skipping the proof of lower bound.

Remark

From Theorem 2.6, $3n+1 \leq \widehat{\mathrm{r}}(P_3, D(n,n)) \leq 3n+4$ if we take $m = n$. Furthermore, we will show $\widehat{\mathrm{r}}(P_3, D(n,n)) = 3n+4$ for $n \geq 5$. For a maximum matching M of a graph F, we denote

$$V(M,n) = \{v \in V(F) \mid \deg(v) \geq n+1 \text{ and } v \text{ covered by } M\}.$$

Then we have the following lemma and theorem.

$\widehat{r}(P_3, D(n,n))$

Lemma 5

Let F be a graph with $e(F) \leq 3n+3$ and $\Delta(F) \geq n+1$, where $n \geq 5$. Let M be a maximum matching of F such that $|V(M,n)|$ is maximized. For any two vertices u, v of degree at least $n+1$, if $uv \in E(F)$, then M covers both u and v.

Theorem 2.7

For $n \geq 5$,

$$\widehat{r}(P_3, D(n,n)) = 3n+4.$$

Skipping the proof.

$\widehat{\mathrm{r}}(P_3, D(1, n))$

Theorem 2.8

For $n \geq 4$,

$$\widehat{\mathrm{r}}(P_3, D(1, n)) = n + 5.$$

Skipping the proof.

$\widehat{r}(P_3, B(3, n))$

Theorem 2.9

For $n \geq 5$,

$$\widehat{r}(P_3, B(3, n)) = n + 6.$$

Upper bound of $\widehat{r}(P_3, B(3, n))$ and $\widehat{r}(P_3, B(4, n))$

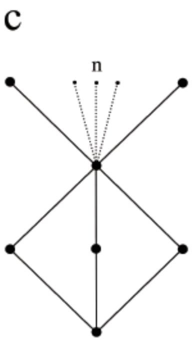


Figure 3: Graphs for Theorem 2.9 and Theorem 2.10.

Lower bound of $\hat{r}(P_3, B(3, n))$

Next we show $\hat{r}(P_3, B(3, n)) \geq n + 6$. Let F be a graph with at most $n + 5$. We can assume that $e(F) = n + 5$. Let s be the number of vertices of degree $\geq n + 1$.

Claim 6

$$s = 1.$$

Proof.

Assume, to the contrary, that $s \geq 2$. Then there exist two vertices, say u_1, u_2, such that $deg_F(u_i) \geq n + 1$ for $i = 1, 2$, and hence there are at least $2(n + 1) - 1 \geq n + 6 > e(F)$, since $n \geq 5$, a contradiction. ∎

Lower bound of $\widehat{r}(P_3, B(3, n))$

Assume $\deg(v) = \Delta(F)$.

- $\Delta(F) \leq n$: no $B(3, n)$ copy.
- $\Delta(F) = n + 1$: choose an edge e incident to v in F and color it red, and the other edges in $F - e$ are colored blue.
- $\Delta(F) \geq n + 3$: color one edge which not incident v red, and the other edges blue.
- $\Delta(F) = n + 2$: need further discussion.

Lower bound of $\widehat{\mathrm{r}}(P_3, B(3, n))$

Suppose $\Delta(F) = n + 2$ and $\deg(v) = \Delta(F)$.

- $F - v \ncong K_3$ and $F - v \ncong K_{1,3}$: there is a $2P_2$ in $F - v$, and we color $2P_2$ red, and the other edges of F are colored blue..
- $F - v \cong K_3$: need further discussion.
- $F - v \cong K_{1,3}$: need further discussion.

Lower bound of $\widehat{r}(P_3, B(3, n))$

Suppose $F - v \cong K_3$ or $F - v \cong K_{1,3}$. Let $N(v) = \{v_i \,|\, 1 \leq i \leq n+2\}$. The center vertex of $K_{1,3}$ is denoted as u, and the other three degree vertices in $K_{1,3}$ are denoted as u_1, u_2 and u_3.
Since F is connected, $|N(v) \cap V(F - v)| \neq \emptyset$.
If $|N(v) \cap V(F - v)| = 1$, let $\{x\} = N(v) \cap V(F - v)$, then we color the edge vx red, the other edges in F blue. One can easily check that there is neither a red copy of P_3 nor a blue copy of $B(3, n)$.

Lower bound of $\widehat{\mathrm{r}}(P_3, B(3, n))$

Case 1

$$F - v \cong K_3$$

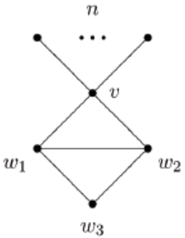
Subcase 1.1

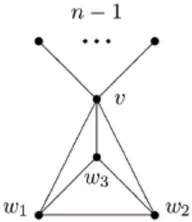
If $|N(v) \cap V(F - v)| = 2$*, then we color the edge* vv_i $(v_i \in N(v)$ *and* $\deg(v_i) = 1)$ *red, the other edges in* F *blue. Then* $v(F - v_i) = n + 3 < v(B(3, n))$*, so there is neither a red copy of* P_3 *nor a blue copy of* $B(3, n)$*.*

Subcase 1.2

If $|N(v) \cap V(F - v)| = 3$*, then* $v(F) = n + 3 < v(B(3, n))$*, which implies no* $B(3, n)$ *copy.*

Lower bound of $\widehat{r}(P_3, B(3, n))$





Lower bound of $\widehat{r}(P_3, B(3, n))$

Case 2

$$F - v \cong K_{1,3}$$

Subcase 2.1

If $N(v) \cap V(F - v) = \{u, u_1\}$*, then we color the edges* uu_2 *and* vv_i $(v_i \in N(v)$ *and* $\deg(v_i) = 1)$ *red, the other edges in* F *blue. Then* $v(F - u_2 - v_i) = n + 3 < v(B(3, n))$*, so there is neither a red copy of* P_3 *nor a blue copy of* $B(3, n)$*.*

Subcase 2.2

If $N(v) \cap V(F - v) = \{u_1, u_2\}$*, then we color the edges* uu_3 *and* vv_i $(v_i \in N(v)$ *and* $\deg(v_i) = 1)$ *red, the other edges in* F *blue. Then* $v(F - u_3 - v_i) = n + 3 < v(B(3, n))$*, so there is neither a red copy of* P_3 *nor a blue copy of* $B(3, n)$*.*

Lower bound of $\widehat{r}(P_3, B(3, n))$

Subcase 2.3

If $|N(v) \cap V(F - v)| = 3$, *then we color the edge* vv_i $(v_i \in N(v)$ *and* $\deg(v_i) = 1$ *red, the other edges in* F *blue. Then* $v(F - v_i) = n + 3 < v(B(3, n))$, *so there is neither a red copy of* P_3 *nor a blue copy of* $B(3, n)$.

Subcase 2.4

If $|N(v) \cap V(F - v)| = 4$, *then* $v(F) = n + 3 < v(B(3, n))$, *which implies no* $B(3, n)$ *copy.*

$\widehat{\mathrm{r}}(P_3, B(4, n))$

Theorem 2.10

For $n \geq 7$,

$$\widehat{\mathrm{r}}(P_3, B(4, n)) = n + 8.$$

Skipping the proof.

Problem of $\widehat{\mathrm{r}}(P_3, B(m, n))$

A good upper and lower bound.

Problem 2.2

For $n \geq ?$, $m \geq ?$,

$$? \leq \widehat{\mathrm{r}}(P_3, B(m, n)) \leq ?.$$

The End

Thank you for your attention !